Title: Exploring Exponential Growth and Decay Functions

Brief Overview:

In this unit, students will use and apply exponential functions of growth and decay in real-world situations. Students will collect and analyze data with the use of the TI-83 graphing calculator, graphs, tables, and other sources. From their research students will formulate predictions and communicate their findings throughout the activities and the performance assessment.

Links to NCTM Standards:

- **Mathematics as Problem Solving**
Students will demonstrate the ability to use problem solving approaches to investigate and understand exponential functions. Students will formulate mathematical models of real-world situations.

- **Mathematics as Communication**
Students will communicate the mathematical concepts of exponential functions through a performance assessment. Students will extend and apply the exponential concepts to real-world situations through newspapers, Internet, and/or phone research.

- **Mathematics as Reasoning**
Students will deduce the validity of growth or decay in real-world situations. Students will construct logical arguments about future implications of given data.

- **Mathematical Connections**
Students will investigate the connections between exponential functions and real-world applications. Students will use technology and statistics to model real-life statistics and data. Students will recognize the algebraic, numerical, and graphical representations of exponential functions.

- **Algebra**
Students will develop and use an equation to make predictions. Students will use tables and graphs as tools to interpret equations.

- **Functions**
Students will model real-world phenomena with decay and growth functions. Students will represent and analyze relationships using tables, graphs, and equations. Students will recognize that a variety of problem situations can be modeled by the same type of function.

- **Statistics**
Students will research and analyze data. Students will construct and draw inferences from charts, tables, and graphs that summarize data from real-world situations.

Links to Maryland High School Mathematics Core Learning Goals

- **1.1.1**
The student will recognize, describe, and extend patterns and functional relationships that are expressed numerically, algebraically, and geometrically.

- **1.1.2**
The student will represent patterns and functional relationships in a table, as a graph, and/or by mathematical expressions.

- **1.2.4**
  The student will describe how the graphical model of a non-linear function represents a given problem and will estimate the solution.

- **3.1.1**
  The student will design and/or conduct an investigation that uses statistical methods to analyze data and communicate results.

- **3.2.1**
  The student will make informed decisions and predictions based upon the results of simulations and data from research.

Grade/Level:

Grades 9-12; Algebra I, Algebra II

Duration/Length:

Four to five class periods (variable)

Prerequisite Knowledge:

Students should have working knowledge of the following skills:

- Use of the TI-83 graphing calculator
- Use of exponents
- Use of order of operations in equations with exponents
- Evaluation and interpretation of graphs and tables

Objectives:

Students will:

- model exponential growth and decay problems.
- analyze and predict future values of exponential functions.
- communicate knowledge of exponential functions in real-world situations.

Materials/Resources/Printed Materials:

- Activity sheets and performance assessment
- Graphing calculator (TI-83)
- TI-83 overhead projector
- Access to research materials

Development/Procedures:

- Start the lesson comparing the graphs of $y = 2x$, $y = x^2$ and $y = 2^x$.
- Students should use the suggested window where recommended.
- After students have completed Activity 1, they should compare results with other classmates. Come together as a class to discuss correct answers.
- Begin Activity 2 with discussion of the formula $y = ab^x$ in real-world situations.
- After students have completed Activity 2, they should understand the effect of a, b, and x.

- For Activity 3 students will need time to research interest rates before they can complete the assignment.
- For Activity 4, students will work in groups of 3 or 4 and produce an exponential decay model. Students will follow TI-83 directions on worksheet to compare their data to the mathematical model.
- After completing all activities, students should be prepared to complete the performance assessment.

Extension/Follow Up:

Contact the county/country from where you were born. Ask for the population from 1900 - 2000 at 10 year intervals. Graph data.

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Activity 1 $y = ab^x$

Use your TI-83 graphing calculator to graph the following functions and answer the questions.

Suggested Window:

```
WINDOW
 Xmin=-10
 Xmax=10
 Xscl=1
 Ymin=-5
 Ymax=20
 Yscl=1
 Xres=1
```

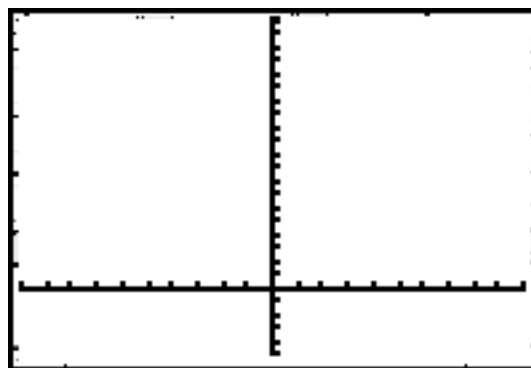A. Graph the following exponential equation

1. $y = 2^x$

a. y-intercept _____

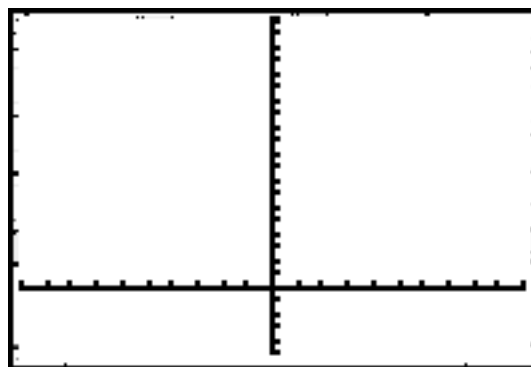b. increasing, decreasing or neither
(circle one)

c. a = _____

d. b = _____

2. $y = 3^x$

a. y-intercept _____

b. increasing, decreasing, or neither
(circle one)

c. a = _____

d. b = _____
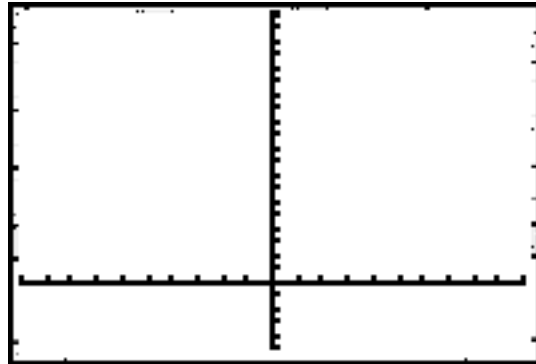
3. $y = 10^x$

a. y-intercept _____

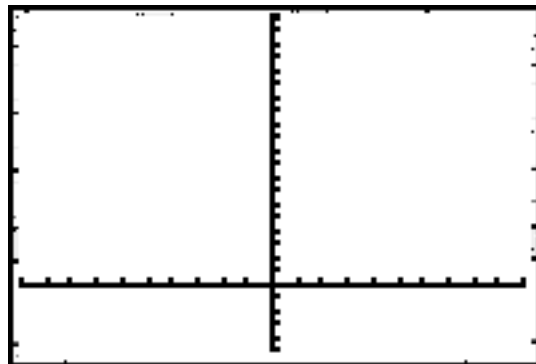b. increasing, decreasing, or neither
(circle one)

c. a = _____

d. b = _____

4. $y = 1^x$

a. y-intercept _____

b. increasing, decreasing, or neither
(circle one)

c. a = _____

d. b = _____
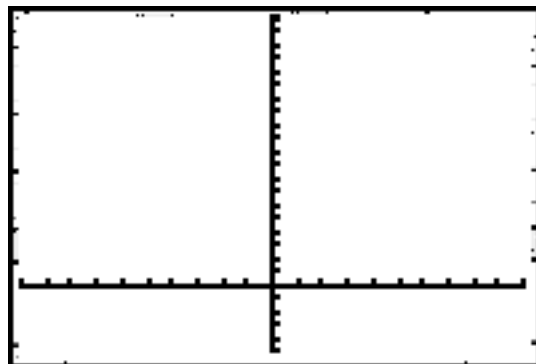
5. $y = (1/2)^x$

a. y-intercept _____

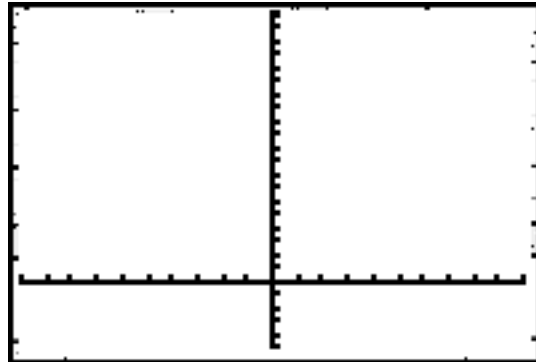b. increasing, decreasing, or neither
(circle one)

c. a = _____

d. b = _____

6. $y = (1/3)^x$

a. y-intercept _____

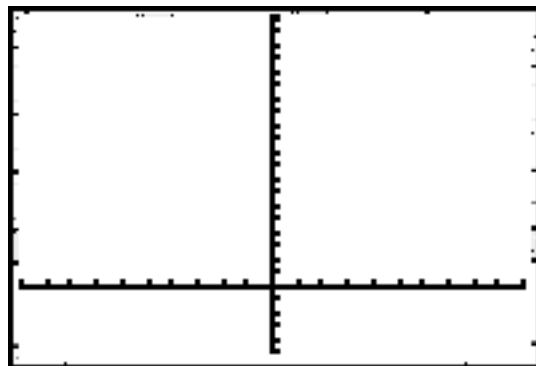b. increasing, decreasing, or neither
(circle one)

c. a = _____

d. b = _____

7. $y = (1/10)^x$

a. y-intercept _____

b. increasing, decreasing, or neither
(circle one)

c. a = _____

d. b = _____

8. What point do they all have in common?_____

9. List the equations that are increasing._____

_____

10. List the equations that are decreasing._____

_____

11. When an exponential graph is increasing , it shows exponential growth .
What are the similarities of all the equations that produce such graphs?
_____
_____

12. When an exponential graph is decreasing , it shows exponential decay. What are the similarities of all the equations that produce such graphs?

_____
_____

13. Are there any graphs which were not reflecting exponential growth or decay? Why? _____
_____

<u>Activity 2</u> $y = ab^x$

1. Sam has $300 to deposit into an account at 6% interest compounded annually. Given the exponential equation $y = 300(1.06)^x$, fill in the table for the future values:

   $y = 300(1.06)^x$

x	years	0	1	2	5	10	20	25
y	$							

a. Where did b=1.06 come from?_____

b. If at the end of year 2 Sam adds $200 to his account, how much will he have in more 10 years?

                                             Answer _____

2. Country A has a declining population. In 1980, the population was 200,000 and it decreases at 3% a year. Given $y = 200,000(.97)^x$, fill in the table: **remember, x = 0 for 1980.**

Year	1980	1981	1982	1985	1990	2000
x						
y						

a. Where did b=.97 come from? _____

b. What was the population in 1970?_____

3. Write an exponential equation, given an initial amount of 500 and a increase of 7%. _____ decrease of 2%_____

4. Ana won $100 and deposited it into a savings account earning 3% interest compounded annually. Write a model for this exponential function. _____

Is it a growth or a decay? _____

a. How much will she have after 5 years?

b. When will her money triple?

5. A new car costs $20,000. It depreciates in value at a rate of 16% each year. Write a model for this exponential function. _____

a. How much will the car be worth after 3 years?

b. How long will it be until the car is worth 1/4 of it's original price?

Activity 3

For this activity, you will be required to do some research.

Your great spinster aunt is very concerned that you will not remember her when she is gone. However, she is also very frugal. She is willing to give you some money to open a savings account. Her condition in giving you this money is that you find the bank that will give you the highest interest and you cannot make any withdrawals for 10 years.

1. How much money did your aunt give you? _____

2. Contact at least 3 financial institutions (either by phone, by looking in the newspaper, or looking on the internet). For this assignment, assume that the given interest rates are compounded annually.

	Financial Insitution	Reference	Interest Rate
1			
2			
3			

3. Choose the bank which will give you the highest interest rate and write the mathematical model. _____

a. How much money will you have in the bank in 20 years?

b. When will your money double?

c. In year 10 you had to withdraw your initial deposit for a down payment on a house. What will the value be in year 20 now?

Activity 4:   The Penny Experiment

Materials: 1 roll of pennies per group
              Box or can
              TI-83 Graphing Calculator

Step 1: Working in groups, open roll of pennies and pour all 50 pennies onto a flat surface. Spread the pennies out and take out all the "heads up" pennies. Write in the table how many pennies you had remaining.

Step 2: Collect those that remain into your box or can, shake them up, and repeat.

Step 3: Repeat step 2 until you have eliminated all the pennies.

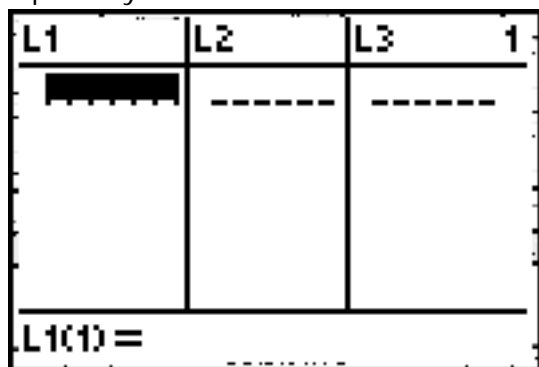| Toss #	Pennies Remaining
0	50

Once you have all the data, follow the directions to use the TI-83 calculator.

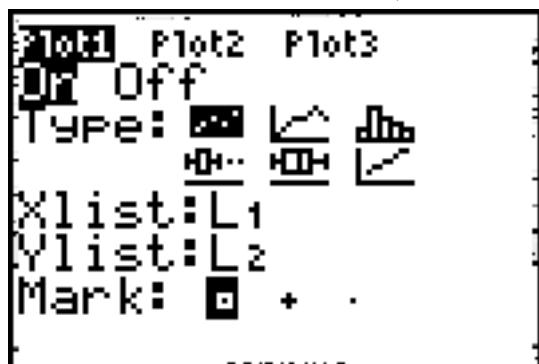Entering Penny Data in TI-83 Graphing Calculator

Press STAT ; Press 1 (Edit)
Begin entering data in L_1 (toss#) and then in L_2 (pennies remaining).
*The last entry should be omitted from the data set. This data point is invalid where y=0 for the equation $y = ab^x$.
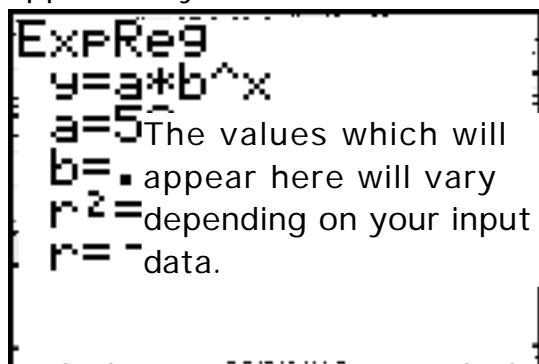
L1	L2	L3	1
▬	------	------	
L1(1)=			

Press 2nd STAT PLOT ; Press 1 (Plot 1); Put cursor on ON and enter.

Plot1 Plot2 Plot3
On Off
Type: 📊 📈 📊
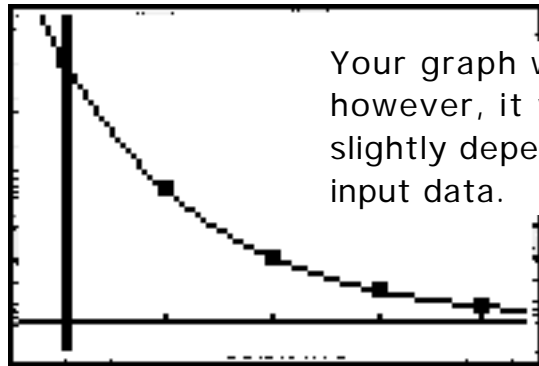   📊 📊 📈
Xlist:L₁
Ylist:L₂
Mark: ▫ + ·

Press ZOOM; Press 9 (ZoomStat) and graph will appear.
To find the best fit graph:
Press STAT ; Press CALC ; Press 0 (ExpReg); The word "ExpReg" will appear on your screen. Press ENTER . You will see the following screen:

ExpReg
 y=a*b^x
 a=5 The values which will
 b=. appear here will vary
 r²= depending on your input
 r= ⁻ data.

Press Y=; Press VARS ; Press 5 (Statistics); arrow over to EQ and Press 1(RegEq). This will return you to the Y= screen with an exponential equation. Press GRAPH .

Your graph will look similar;
however, it will vary
slightly depending on your
input data.

1. What is your exponential equation? (Look back at the Y= and round to the nearest hundredth). _____

2. What is the mathematical model for this exponential decay? _____

3. Explain why your equation does/does not match the mathematical model.
_____
_____
_____

4. What would allow your exponential equation to approach the mathematical model? _____
_____

Activity 1 $y = ab^x$ **KEY**

Use your TI-83 graphing calculator to graph the following functions and answer the questions.

```
WINDOW
 Xmin=-10
 Xmax=10
 Xscl=1
 Ymin=-5
 Ymax=20
 Yscl=1
 Xres=1█
```
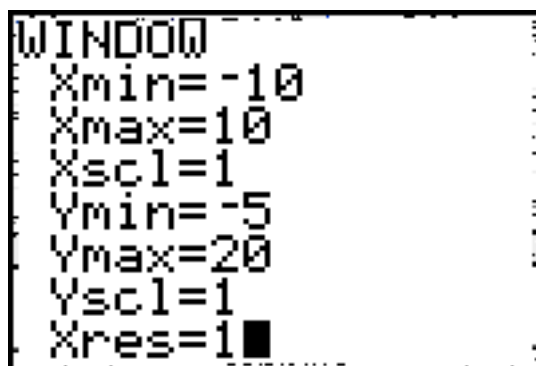
A. Graph the following exponential equation

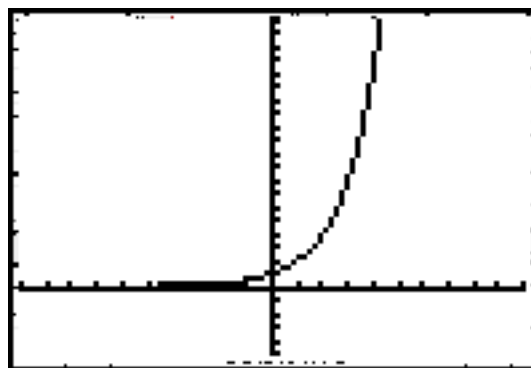1. $y = 2^x$

a. y-intercept __(0, 1)__

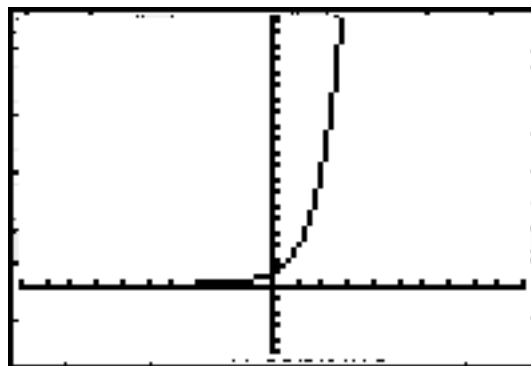b. increasing , decreasing, or neither (circle one)

c. a = ___1___

d. b = ___2___

2. $y = 3^x$

a. y-intercept __(0, 1)__

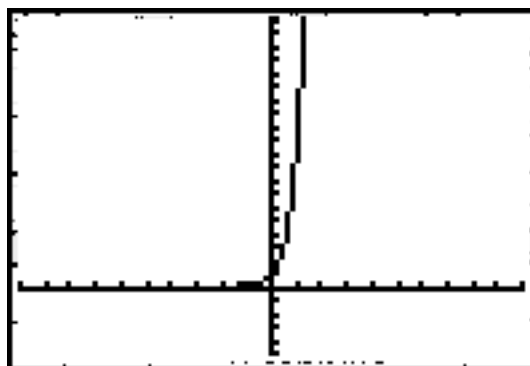b. increasing, decreasing or neither (circle one)

c. a = ___1___

d. b = ___3___

3. $y = 10^x$

a. y-intercept __(0, 1)__

b. increasing , decreasing, or neither (circle one)

c. a = ___1___

d. b = ___10___

4. $y = 1^x$

a. y-intercept __(0, 1)___

b. increasing, decreasing, or neither (circle one)

c. a = ___1___

d. b = ___1___

5. $y = (1/2)^x$

a. y-intercept __(0, 1)__

b. increasing, decreasing , or neither (circle one)
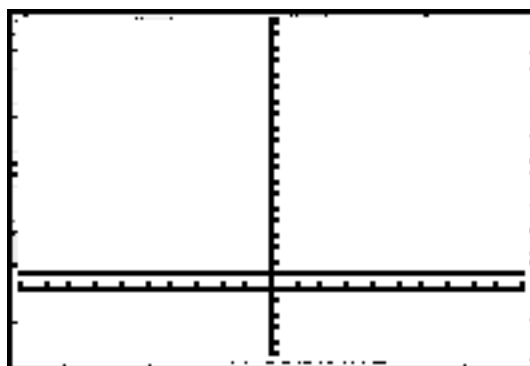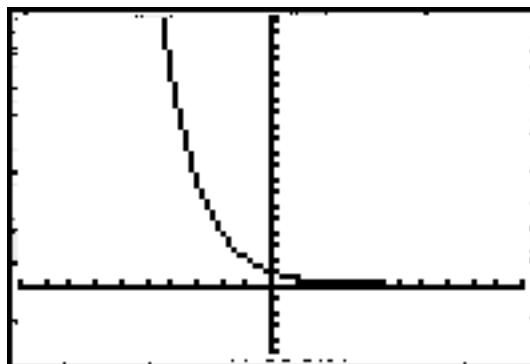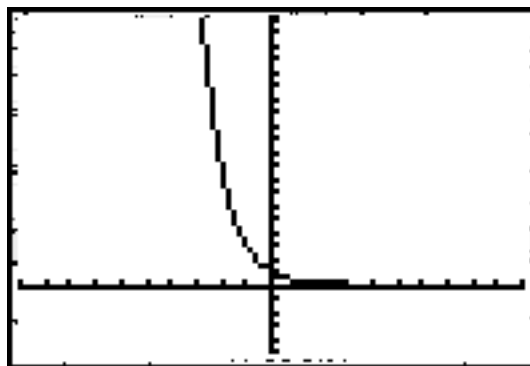
c. a = ___1___

d. b = ___1/2___

6. $y = (1/3)^x$

a. y-intercept __(0, 1)__

b. increasing, decreasing , or neither (circle one)

c. a = ____1____

d. b = ____1/3____

7. $y = (1/10)^x$

a. y-intercept __(0, 1)__

b. increasing, decreasing , or neither (circle one)

c. a = ____1____

d. b = ____1/10____
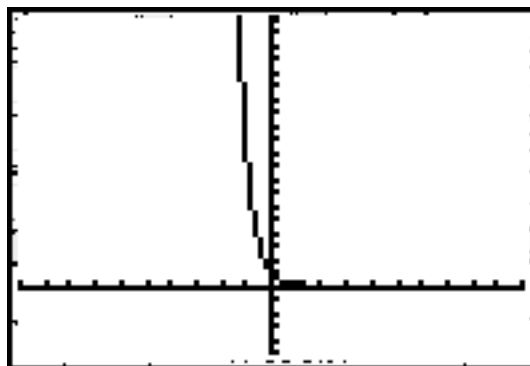
8. What point do they all have in common?___y-intercept _(0,1) __

9. List the equations that are increasing.____$y=2^x$, $y=3^x$, $y=10^x$_____

_____

10. List the equations that are decreasing._$y=(1/2)^x$, $y=(1/3)^x$, $y=(1/10)^x$

11. When an exponential graph is increasing , it shows exponential growth . What are the similarities of all the equations that produce such graphs? ___b>1____the base is greater than 1   _____
_____

12. When an exponential graph is decreasing , it shows exponential decay. What are the similarities of all the equations that produce such graphs? ____0<b<1____base is between zero and one   _____

_____

13. Are there any graphs which were not reflecting exponential growth or decay? Why? _____$y=1^x$_____1 to any power will be 1, producing a horizontal line at y=1   _____

<u>Activity 2</u> $y = ab^x$

1. Sam has $300 to deposit into an account at 6% interest compounded annually. Given the exponential equation $y = 300(1.06)^x$, fill in the table for the future values:

$y = 300(1.06)^x$

x	years	0	1	2	5	10	20	25
y	$	300	318	337.08	401.47	537.25	962.14	1287.56

a. Where did b=1.06 come from?_____1+.06 _____

b. If at the end of year 2 Sam adds $200 to his account, how much will he have in 10 more years?

$537.08(1+.06)^{10}$

Answer _$961.83 _____

2. Country A has a declining population. In 1980, the population was 200,000 and it decreases at 3% a year. Given $y = 200,000(.97)^x$, fill in the table: **remember, x = 0 for 1980.**

Year	1980	1981	1982	1985	1990	2000
x	0	1	2	5	10	20
y	200,000	194,000	188,180	171,747	147,485	108,759

a. Where did b=.97 come from? _____1-3%, 1-.03   _____

b. What was the population in 1970?_____271,214 _____

$200,000(.97)^{-10}$

3. Write an exponential equation, given an initial amount of 500 and a increase of 7%. ____$y=500(1.07)^x$____ decrease of 2%__$y=500(.98)^x$__

4. Ana won $100 and deposited it into a savings account earning 3% interest compounded annually. Write a model for this exponential function. _____$y=100(1.03)^x$_____
Is it a growth or a decay? _____growth _____

a. How much will she have after 5 years?

$$y=100(1.03)^5 \qquad\qquad \$115.93$$

b. When will her money triple?

37 years: $298.52; 38 years: $307.48

5. A new car costs $20,000. It depreciates in value at a rate of 16% each year. Write a model for this exponential function. ___$y=20,000(.84)^x$____

a. How much will the car be worth after 3 years?

$$\$11,854.08$$

b. How long will it be until the car is worth 1/4 of it's original price?

1/4 original price = $5,000

$20,000(.84)^7 = \$5901.81$

$20,000(.84)^8 = \$4957.52$

<u>Activity 3</u>

For this activity, you will be required to do some research.

Your great spinster aunt is very concerned that you will not remember her when she is gone. However, she is also very frugal. She is willing to give you some money to open a savings account. Her condition in giving you this money is that you find the bank that will give you the highest interest and you cannot make any withdrawals for 10 years.

1. How much money did your aunt give you? ___answers will vary   _____

2. Contact at least 3 financial institutions (either by phone, by looking in the newspaper, or looking on the Interrnet). For this assignment, assume that the given interest rates are compounded annually.

	Financial Insitution	Reference	Interest Rate
1		Answers will vary, but to insure accurate	
2		work, share with your students that the	
3		reference for newspapers needs to be the	

name and date; for phone--name of person spoken with; for internet--web address.

3. Choose the bank which will give you the highest interest rate and write the mathematical model. _____answers will vary   _____

a. How much money will you have in the bank in 20 years?

answers will vary

b. When will your money double?

answers will vary

c. In year 10 you withdraw your initial deposit for a down payment on a house. What will the value be in year 20 now?

answers will vary

Activity 4:   The Penny Experiment

Materials: 1 roll of pennies per group
              Box or can
              TI-83 Graphing Calculator

Step 1: Working in groups, open roll of pennies and pour all 50 pennies
onto a flat surface. Spread the pennies out and take out all the "heads up"
pennies. Write in the table how many pennies you had remaining.

Step 2: Collect those that remain into your box or can, shake them up, and
repeat.

Step 3: Repeat step 2 until you have eliminated all the pennies.

Toss #	Pennies Remaining
0	50
Answers will vary.	

Once you have all the data, follow the directions to use the TI-83
calculator.

Entering Penny Data in TI-83 Graphing Calculator

Press STAT ; Press 1 (Edit)
Begin entering data in L_1 (toss#) and then in L_2 (pennies remaining).
*The last entry should be omitted from the data set. This data point is invalid where y=0 for the equation $y = ab^x$.

L1	L2	L3	1
▮▮▮▮▮▮	------	------	

L1(1)=

Press 2nd STAT PLOT ; Press 1 (Plot 1); Put cursor on ON and enter.

Plot1 Plot2 Plot3
On Off
Type: ▦ ⬥ ⬚
Xlist:L₁
Ylist:L₂
Mark: ▢ + ·

Press ZOOM; Press 9 (ZoomStat) and graph will appear.
To find the best fit graph:
Press STAT ; Press CALC ; Press 0 (ExpReg); The word "ExpReg" will appear on your screen. Press ENTER . You will see the following screen:

ExpReg
 y=a*b^x
 a=50
 b=.493914043
 r²=.   answers will   78
 r=-.   vary   76

Press Y=; Press VARS ; Press 5 (Statistics); arrow over to EQ and Press 1(RegEq). This will return you to the Y= screen with an exponential equation. Press GRAPH .
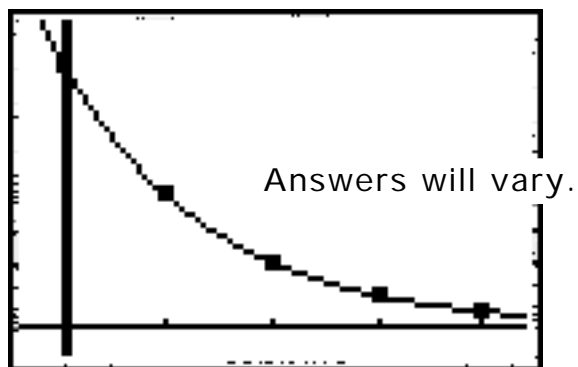
Answers will vary.

1. What is your exponential equation? (Look back at the Y= and round numbers to the nearest hundredth). ___answers will vary   _____

2. What is the mathematical model for this exponential decay?
_____y=50(.5)^X_____

3. Explain why your equation does/does not match the mathematical model. _Although the probability of tossing a head is 1/2, we have only done 1 trial and therefore our pennies could turn out in any pattern. _____

4. What would allow your exponential equation to approach the mathematical model? __repeated trials and average each toss   _____

Teacher's Notes for Activities

Before you begin this unit, there are a few definitions and concepts you will want to review with your students:

- mathematical model
- $y = ab^X$ and order of operations with exponents (specifically review zero power)
- compounded annually : **note that all problems in this unit will be using interest that is compounded annually**; remember that we are using the formula, amount = principal $(1 + interest)^X$
- depreciate
- appreciate

Activity 1

- It might be helpful to discuss with your students values of a that vary from 1 and how this would affect the graphs (due to the fact that you will be using real world applications which often have an a value greater than 1).

Activity 2

- One of the purposes of this activity is for students to discover the relationship between growth/decay and the b value (1+% is growth;1-% is decay).
- Consider encouraging students to use the table function on your TI-83 to find the answers for the problems involving tables.

Activity 3 & 4

- For these two activities, the answers the students/groups generate will vary depending on their input data.
- It might be fun to have an extended discussion with your students about the depreciation value of a car. How legitimate can an answer for the value in 25, or even 50 years be? Consider this discussion: At some point an individual car will either be destroyed for parts or will become an antique. Which models of cars are more/less likely to become antiques? When is that likely to happen?

Performance Assessment

Teacher's Guide

Introduction

The purpose of this Assessment Activity is to provide feedback so that appropriate instructional decisions can be made. The task assesses the application of objectives that are included in the exponential function unit.

Objectives Covered

This task assumes that your students have received instruction and have been assessed on the following concepts and/or objectives:

- modeling exponential and decay problems.
- analyzing and predicting future values of exponential functions.
- communicating knowledge of exponential functions in real world situations.

Tools/Materials Needed for Assessment
- Pencils, paper
- TI-83 graphing calculator
- Copies of assessment for all students

Administering the Assessment

This task should take approximately 50 minutes to administer.

Distribute a copy of the assessment to each student.

SAY: Today you are going to use your mathematics skills to complete items in this assessment. Use your TI-83 graphing calculator when necessary. You will complete everything on your own. Take time now to skim through the different parts of the assessment.

Pause to allow students to look through the assessment. Have the students write their name on the first page.

SAY: Are there any questions? You may begin.

When students have completed the assessment, collect their materials.

Student Response Sheet

Exponential Growth and Decay Functions

1. Identify the following equations as exponential growth or decay.

    a. $y = 3^x$ _____

    b. $y = (1/2)^x$ _____

    c. $y = 2000(.97)^x$ _____

    d. $y = 20(1 + .02)^x$ _____

    e. $y = 400(1.06)^x$ _____

    f. Contrast the characteristics of an equation of exponential growth versus exponential decay.

2. Your dream car costs $42,000. You find that depreciation is 16% per year.

    a. Fill out the table using the formula $42000(1-.16)^x$ to find the car's value in future years.

years	0	2	4	6	8	10
value						

    b. If you could only afford a $20,000 car, how old would the car be when you could afford it? Explain how you determined the car's age.

3. The population of Homeword County was 16,700 in 1900 and increased at a rate of 9% per year until 1950.
   a. What is the equation that best fits this situation. List the steps you used to find the equation.

   b. Fill out the table showing the population growth every 10 years.

   HOMEWORD COUNTY

| year	1900					
x						
population						

   c. If during the next 10 years there is a population boom and the population increases at a rate of 15%, find the population for 1960.

      a. 2,939,723
      b. 2,872,292
      c. 5,023,656
      d. 5,443,922,506

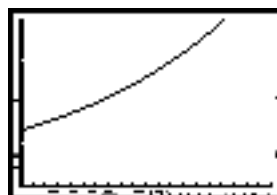4. The artwork of an aging artist is appreciating at a rate of 7% a year. A new painting costs you $2000.
   a. Write an equation for the given situation.

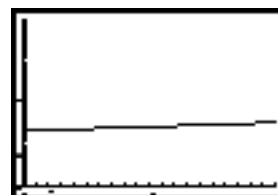   b. Using your graphing calculator, graph your equation and choose the best graph.

a.

```
WINDOW
 Xmin=0
 Xmax=20
 Xscl=1
 Ymin=0
 Ymax=6000
 Yscl=100
 Xres=1■
```

b.

c.
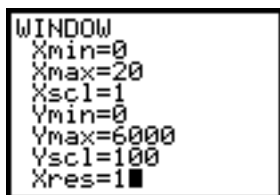
   c. Using your table function, how much is your painting worth after 5 years?

d. When will your painting have doubled in value?

e. If the artist dies after 5 years and because of his death his artwork now appreciates at 20% a year, how much will it be worth after 10 more years?

    a. $12,383.47
    b. $30,814.04
    c. $6,979.99
    d. $17,368.44

f. Fifteen years after you purchased the painting, it's discovered that the artist is a phony and his work now depreciates at a rate of 50% per year. What is the value of your painting after 15 more years? Explain how you determined the painting's value.

Performance Assessment

   Scoring Guide

Exponential Growth and Decay Functions

1. Identify the following equations as exponential growth or decay.

   a. $y = 3^x$ __growth _____

   b. $y = (1/2)^x$ __decay _____

   c. $y = 2000(.97)^x$ __decay _____

   d. $y = 20(1 + .02)^x$ __growth _____

   e. $y = 400(1.06)^x$ __growth _____

   f. Contrast the characteristics of an equation of exponential growth versus exponential decay.

     For the exponential equation of the form $y = ab^x$
     1. b greater than one - growth
     2. b greater than zero but less than one - decay
     3. a greater than zero

     Rubric
       3 Explain answer fully with all three characteristics
       2 Explain answer with one characteristic missing
       1 One characteristic explained
       0 no work shown or no correct characteristics

2. Your dream car costs $42,000. You find that depreciation is 16% per year.

   a. Fill out the table using the formula $42000(1-.16)^x$ to find the car's value in future years.

years	0	2	4	6	8	10
value	$42,000	$29,635	$20,911	$14,755	$10,411	$7346

b. If you could only afford a $20,000 car, how old would the car be when you could afford it? Explain how you determined the car's age.

Answer - 5 years old

Rubric

3 - answer correct with full accurate explanation
2 - answer correct with some accurate explanation or answer incorrect with accurate explanation
1 - answer incorrect with some accurate explanation
0 - answer incorrect with no explanation

3. The population of Homeword County was 16,700 in 1900 and increased at a rate of 9% per year until 1950.
   a. What is the equation that best fits this situation. List the steps you used to find the equation.

   Answer - $y = 16,700(1.09)^x$

   Rubric

   3 - all steps shown and an accurate equation
   2 - some steps shown and an accurate equation
   1 - some steps shown and inaccurate equation or accurate equation and no steps shown
   0 - no steps shown and inaccurate equation

   b. Fill out the table showing the population growth every 10 years.

   HOMEWORD COUNTY

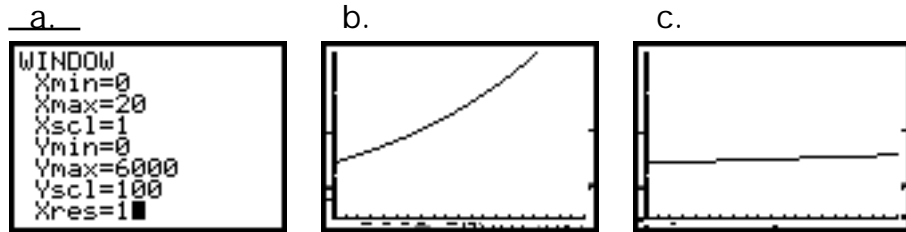| year	1900	1910	1920	1930	1940	1950
x	0	10	20	30	40	50
population	16,700	39,535	93,594	221570	524537	1241771

   c. If during the next 10 years there is a population boom and the population increases at a rate of 15%, find the population for 1960.

      a. 2,939,723
      b. 2,872,292
      c. 5,023,656
      d. 5,443,922,506

4. The artwork of an aging artist is appreciating at a rate of 7% a year. A new painting costs you $2000.

   a. Write an equation for the given situation.

   $$y = 2000(1.07)^x$$

   b. Using your graphing calculator, graph your equation and choose the best graph.

   <u>a.</u>   b.   c.

   ```
   WINDOW
    Xmin=0
    Xmax=20
    Xscl=1
    Ymin=0
    Ymax=6000
    Yscl=100
    Xres=1■
   ```

   c. Using your table function, how much is your painting worth after 5 years?

   $2805.10

   d. When will your painting have doubled in value?

   11 years

   e. If the artist dies after 5 years and because of his death his artwork now appreciates at 20% a year, how much will it be worth after 10 more years?

   a. $12,383.47
   b. $30,814.04
   c. $6,979.99
   d. <u>$17,368.44</u>

f. Fifteen years after you purchased the painting, it's discovered that the artist is a phony and his work now depreciates at a rate of 50% per year. What is the value of your painting after 15 more years? Explain how you determined the painting's value.

Answer - $0.53

Rubric

3 - answer correct with full accurate explanation

2 - answer correct with some accurate explanation or
    answer incorrect with accurate explanation

1 - answer incorrect with some accurate explanation

0 - answer incorrect with no explanation